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JEL Classification O40, Q50, E10, E22

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Continuous vs Discrete Time Modelling in Growth and Business Cycle Theory*

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1 Introduction

The time dimension is of fundamental importance for macroeconomic theory, since most macroeconomic problems deal with intertemporal trade-offs. In modeling time, economists move from discrete to continuous time according to their methodological needs, as if both ways of representing time were equivalent. For example, growth theory is mainly written in continuous time, but business cycle theory is in a large extend written in discrete time. However, they refer to each other as being two pieces of the same framework.

The view that continuous and discrete time representations are equivalent is mainly supported by limit properties: the discrete time version of the standard dynamic general equilibrium model does converge to its continuous time representation when the period length tends to zero. However, this view hides a fundamental problem of timing. In continuous time, investment at time t becomes capital at time $t + \mathrm{d}t$. The discrete time equivalent is that period t investment transforms into capital at period $t+1$. Thus, the speed at which investment becomes capital depends directly on the length of the period, and this is of fundamental importance as far as one deals with inter-temporal trade-offs. Also, there is the issue of self-fulfilling prophecies in dynamic equilibrium models for which business cycle fluctuations may be driven by beliefs or *animal spirits*. A *sunspot* shock can be defined over the parametric space for local indeterminacy of equilibria, which in its turn may critically depend on the time dimension as far as we set empirically plausible parameterizations for the model. Our discussion in this paper is mostly related to these two issues in macroeconomic dynamics.

Several authors have exploited these fundamental differences to study the properties of discrete versus continuous time models. The classical references discuss differences between discrete and continuous time representations arising from uncertainty [cf. Burmeister and Turnovsky (1977)] or adjustment costs [cf. Jovanovic (1982)]. Later on, Carlstrom and Fuerst (2005) focus on determinacy in monetary models, or Hintermaier (2003) and Bambi and Licandro (2005) on the dependency of the conditions for indeterminacy on the frequency of the discrete time representation of the model of Benhabib and Farmer (1994). Key ingredients for local indeterminacy tipically relate to some form of market friction such as either imperfect competition (increasing returns to scale in production or market power in trade) or externalities (own production or consumption depends on other agents' in the same or the other side of the market).

Finally, there is a literature that looks to models in continuous time with discrete elements. Benhabib (2004) builds upon distributed lag structures to make the pure continuous time and discrete time frameworks emerge as special cases of a system of differential equations with delays. Anagnostopoulos and Giannitsarou (2005) propose a general continuous time model where certain events take place discretely, whereas Licandro and Puch (2006) use optimal control theory with delays to characterize the gap between discrete and continuous time models. As these authors, we discuss next a general framework where the pure continuous time and discrete-time representations emerge as special cases. Different from them we stress on the unified framework provided by optimal control with delays. See Kolmanovskii and Myshkis (1998), and recent applications by Boucekkine et al (2005), Licandro et al (2008), and others.

Before using optimal control with delays, we introduce continuous and discrete time representations in a standard macroeconomic framework. We start with a baseline description of continuous vs discrete time modelling in growth and business cycle theory. To this purpose we introduce the Solow growth model and the Ramsey model of the business cycle together with a description of the economic equilibrium. This description follows selected sections in Farmer (1999) or Novales et al. (2008). Then we turn to the issue of local indeterminacy in a canonical growth model of the environment which is subject to a pollution externality building upon Fernández et al. (2012). We show with a simple illustration that the time period of the model critically modifies its parameterization, and thus the empirically plausible space for local indeterminacy. This leads to differences in transitional dynamics that are quantitatively meaningful. Finally, with a time-to-build example we show that the discrete time representation of the standard optimal growth model implicitly imposes a particular form of time–to–build to the continuous time representation. Time–to-Build in discrete time is analyzed by Kydland and Prescott (1982). An alternative version with this assumption in continuous time is in Asea and Zak (1999) and Collard et al (2008). Here we show that the discrete time version is a true representation of the continuous time problem under some sufficient conditions.

The organization of the paper is as follows. We start by introducing a simple dynamic system that has proven useful to discuss questions in economic growth theory. Thus, Section 2 describes the Solow growth model in continuous and discrete time and their uses. Then we move forward to the use of optimal control theory and the concept of competitive equilibrium. In so doing we describe the Ramsey model for the business cycle in Section 3, and the description is both in continuous and discrete time. Section 4 discusses some of the consequences of the differences in the time dimension for indeterminacy, and we focus on steady states and transitional dynamics. In Section 5 we propose a time-to-build extension of the theory by using optimal control theory with delays. Section 6 concludes.

2 The Solow growth model

The workhorse of economic growth theory is the *Solow model* in discrete time [cf. Solow (1956)]. The Solow growth model is characterized by the following set of equations:

$$\begin{aligned}
&Y_t = F(K_t, A_t L_t), \\
&K_{t+1} = (1-\delta)K_t + I_t, \quad K_0 = \bar{K}_0, \\
&I_t = s\,Y_t, \quad 0 < s < 1, \\
&A_t = \gamma^t A_0, \quad \gamma \equiv 1+g > 1, \quad A_0 = \bar{A}_0, \\
&L_t = 1, \quad (\text{might be} \quad L_t = \gamma_L^t\, L_0).
\end{aligned} \tag{2.1}$$

The key assumptions are: *i)* savings (equal to investment, I_t) are a constant fraction, s, of output, Y_t, and *ii)* $F(\bullet)$ is a *neoclassical production function* in capital, K_t, and labor, L_t, where A_t represents the state of the production technology. In particular, let $k_t = \frac{K_t}{A_t L_t}$ and $F(\bullet)$ homogenous of degree one, then the equilibrium of the model is described by

$$\gamma k_{t+1} = (1-\delta)k_t + s\, f(k_t), \tag{2.2}$$

a first-order nonlinear difference equation. Note $F(\bullet)$ neoclassical implies $f'(k) > 0$, $f''(k) < 0$, $\forall k > 0$, and we further assume $\lim_{k\to\infty} f'(k) = 0$, $\lim_{k\to 0} f''(k) = \infty$. Figure 1 summarises the equilibrium of the Solow growth model written in efficiency units, k, with $A_0 = 1$ so that $y_t = k_t^\alpha$, corresponds to $Y_t = K_t^\alpha\,(A_t L_t)^{1-\alpha}$.

Even in this simple representation of an economic model with dynamics induced by the accumulation of a stock (of physical capital K_t in this case), there is already an important approximation. Such an approximation comes from the fact that the definition of the capital stock is primarily established from cumulative investment in continuous time and not in discrete time. Therefore:

$$K(t) = \int_{-\infty}^{t} I(s)\ \mathrm{e}^{-\delta(t-s)} ds, \tag{2.3}$$

with the additional simplifying assumption here that investment in different capital vintages can be added up at an exponential price $q(s) \equiv\ \mathrm{e}^{-\delta(t-s)}$, a simplification that is labelled the *law of permanent inventory* and corresponds to the assumption of exponential depreciation at the rate δ.

If this is the case, the accumulation of physical capital in continuous time is described by

$$K'(t) = I(t) - \delta K(t), \tag{2.4}$$

a differential equation whose solution is the integral equation (2.3) above, as it can be obtained from direct differentiation of the integral equation. With (2.4) rather than $K_{t+1} = (1-\delta)K_t + I_t$ one arrives to the continuous time version of the Solow model, with all the rest of the system (2.1) remaining the same as above, provided

$$\begin{aligned} &A(t) = A(0)\,\mathrm{e}^{g\,t}, \quad g > 0, \quad \text{note above } \gamma = (1+g) \\ &L(t) = 1 \quad \text{for instance, and with the assumptions above,} \end{aligned}$$

are defined correspondingly. Consider further again $k(t) = \frac{K(t)}{A(t)\,L}$. Then:

$$k'(t) = s\,f(k(t)) - (\delta + g)\,k(t) \quad (\text{or } \dot{k}_t = s\,f(k_t) - (\delta + g)\,k_t,\ \text{with } \dot{k}_t \equiv k'(t)\ ), \tag{2.5}$$

describes now the equilibrium of the model in continuous time: a first-order nonlinear differential equation. Note the use of the two alternative notations in continuous time. More on this below.

Regarding the solution of the equilibrium representation in (2.5) we know that under $f(k(t)) = k(t)^{\alpha}$, and with defined $z(t) \equiv k(t)^{1-\alpha}$, it is obtained that $z(t) = (z(0) - \bar{z})\,\mathrm{e}^{\phi\,t}, \quad \phi = -(1-\alpha)\,(\delta + g)$, by guessing $z(t) = c\,\mathrm{e}^{\phi\,t} + \bar{z}$, with $\bar{z} = s/(\delta + g)$. Unfortunately there is no exact solution for the discrete time case. Thus, the discrete time assumption involves not only a key approximation in the specification of the model, but also an approximation in obtaining the solution of (2.2) either recursively from an initial condition or by linearizing about the steady state.

All in all, dynamic economic models in continuous time form the basis for economic growth theory, as far as those models are more analytically tractable that their discrete version counterparts. However, the discrete time version of the models has a clear advantage when it comes to the issue bringing the model to the data. The full characterisation of the dynamics in discrete time often involves linear approximation about the steady state solution though. Precisely, the Steady State k_{ss} is found from solving squation (2.2) irrespective of time:

$$(\gamma + \delta - 1)k_{ss} = s\, f(k_{ss}).$$

Notice that if k_t converges to k_{ss}, then K_t converges to a trend. One can then (log-)linearize around k_{ss} to obtain (one may further consider linearization vs log-linearization):

$$\hat{k}_{t+1} \simeq \frac{(1-\delta) + s\, f_k(k_{ss})}{\gamma} \hat{k}_t.$$

where $\hat{k}_t \equiv \left(\frac{k_t - k_{ss}}{k_{ss}}\right) \simeq \log k_t - \log k_{ss}$ and therefore, backward substitution implies (from an approximated convergence result as before)

$$\log k_{t+1} - \log k_{ss} \simeq a^t (\log k_0 - \log k_{ss}),$$

where $a = \frac{(1-\delta)+s\, f_k(k_{ss})}{\gamma}$ ($a < 1$ since $f(\bullet)$ is concave). Figure 2 illustrates again the equilibrium relation, $g(k)$, as it corresponds to representation (2.2) above, together with a linear approximation around its steady state. Richer approximations will allow us to characterize equilibrium dynamics of growth models also in a stochastic environment, and with arbitrary precision. Indeed, one can consider a version of system (2.1) above, but now stochastic with,

$$Y_t = \upsilon_t\, F(K_t, A_t L_t), \quad \upsilon_t \sim D\left[A, B\right],$$

where $D\left[A, B\right]$ represents the probability distribution of υ_t. The equilibrium of the stochastic growth economy is then:

$$\gamma k_{t+1} = (1-\delta)k_t + \upsilon_t\, s\, f(k_t),$$

and $(\gamma + \delta - 1)k_{ss} = \bar{\upsilon}\, s\, f(k_{ss})$ is the steady state, provided $\bar{\upsilon} \equiv E\left[\upsilon_t\right]$. Further parameterization of the shock process υ_t may be required for business cycle purposes. For instance, Figure 3 depicts simulated output under $\theta_t\, k_t^{\alpha}$ with $\theta_t = \bar{\theta}^{1-\rho}\, \theta_{t-1}^{\rho}\, \varepsilon_t$, so that $\log \varepsilon_t \sim_{\text{iid}} N(0, \sigma_{\varepsilon}^2)$ and ρ is the persistence parameter of the process. The example suggests that it make sense to think of fluctuations as caused by shocks to productivity of a neoclassical growth economy, in a richer environment though.

Summarizing, we have presented here the basic framework of the Solow growth model to show that the issue of continuous vs discrete time representation arises. We have also revised the basic methodology to deal with the model and its solution, introducing the issue of linear approximation to nonlinear models in discrete time around stable steady states. The interested readers may refer to Farmer (1999) and Novales et al. (2008) for further details. Next we introduce optimisation and economic equilibrium in the framework of the *Ramsey growth model* as the building blocks of modern business cycle theory, while focusing on the consequences for policy functions and equilibrium determination of changes in the frequency of decisions in the model.

3 The Ramsey model and the business cycle

Rather than assuming that savings are a constant fraction of income as in the Solow model, now it is assumed that there exist a representative household that confronts a consumption-saving trade-off in discrete time. Moreover, it is assumed that such a representative household orders infinite sequences of consumption streams by using a well behaved felicity function given by $U(c_t)$, such that $U'(c_t) > 0,\ U''(c_t) < 0,\ \forall c_t > 0$, according to:

$$\sum_{t=0}^{\infty} \beta^t U(c_t), \ \ \beta \in (0,1),$$

where β is the subjective discount factor of the household, and captures its degree of impatience in discrete time. Again we abstract from population growth, and we consider $L = 1$. Also, we abstract from exogenous technical progress A_t, and we assume that technology frontier is given by some $f(k_t)$, under the regularity conditions above.

Without loss of generality, let us specify the problem of a representative household that takes the decision to invest in physical capital in a centralized framework. Thus,

$$k_{t+1} - k_t = f(k_t) - c_t - \delta k_t, \tag{3.1}$$

and therefore, the decision problem of the *social planner* in discrete time can be written as

$$\max_{\{c_t\,,\,k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t U(c_t) \qquad \text{subject to (3.1), and given } k_0. \tag{3.2}$$

The assumption of a representative household is more general than it first appears, and there are conditions under which the market allocation with many agents can be achieved by a social planner. This result can be established by using the two fundamental theorems of welfare economics [cf. Debreu (1959)]. At this point, however, it just allows us to focus on quantities and abstract from market prices. Later on, when necessary, we will be more precise on the type of market economy that supports the centralized problem at hand. This implies, in the case of the economy we are describing here, that a social planner maximizes the preferences of the representative household subject to the feasibility constraint of the economy.

The key issue here is that the dynamic optimization approach breaks the tight connection between output and savings in the short-run, the one that we had in the Solow growth model. The discrete time framework allows us to bring the model to the data. These two elements have made some elaborated extensions of the model in (3.2) to form the basis for the theory of business cycles. Thus, according to modern business cycle theory, if productivity shocks are persistent and of the right magnitude, business cycle fluctuations are what growth theory predicts. It is the case though, that for a given volatility of the exogenous state, and then of the endogenous state, the volatility of the control variable differs with the frequency of decisions in the model. Next, we illustrate this issue in the basic Ramsey model in continuous vs discrete time.

Let us characterize further the problem (3.2) above. The first-order conditions of this problem are given by:

$$U'(c_t) = \lambda_t,$$

$$\lambda_t = \lambda_{t+1}\beta \left[f'(k_{t+1}) + 1 - \delta\right],$$

$$\lim_{T\to\infty} \lambda_{t+T} k_{t+T+1} = 0,$$

all $t \geq 0$, where λ_t is the shadow price in utility units of a unit of capital at time t. From the first-order conditions, the standard Euler equation condition is obtained:

$$\frac{U'(c_t)}{\beta U'(c_{t+1})} = 1 + r(k_{t+1}) \tag{3.3}$$

where we are assuming that the return to capital is just the extra output that the economy gets from an extra unit of capital, that is $r(k) = \frac{\partial [f(k) - \delta k]}{\partial k} = f'(k) - \delta$.

The continuous time version of the problem basically involves the discount factor, $\beta = 1/(1+\rho)$, where now ρ is the subjective discount rate, and the approximation of first-differences by derivatives, that is,

$$\begin{aligned} &U'(c_{t+1}) - U'(c_t) \simeq U''(c_t)\,\dot{c}_t, \text{ since } \lim_{dt \to 0} \tfrac{U'(c_{t+dt}) - U'(c_t)}{dt} = U''(c_t)\,\dot{c}_t, \\ &k_{t+1} - k_t \simeq \dot{k}_t \ . \end{aligned} .$$

where we are introducing as in (2.5) the notation $\dot{x}_t \equiv x'(t)$ that we will use for the rest of the paper, and that correspondingly substitutes the notation $x(t)$ by the notation x_t to be used both in continuous and discrete time, except for Section 5 below that we combine both $\dot{x}_t$ and $x(t-d)$.

Consequently, the Euler equation Euler (3.3) and the aggregate resource constraint (3.1) are respectively transformed into:

$$\frac{1}{\beta(1+r)} = 1 + \frac{U''(c_t)}{U'(c_t)}\dot{c}_t \quad \underbrace{\rightarrow}_{\frac{1}{\beta(1+r)} = \frac{1+\rho}{1+r} \simeq 1+\rho-r} \quad \frac{U''(c_t)}{U'(c_t)}\dot{c}_t = \rho - r \tag{3.4}$$

$$\dot{k}_t = f(k_t) - c_t - \delta k_t, \tag{3.5}$$

where we assume that ρ and r are "small". Clearly though, the smaller the time period, the worse the approximation. However, given the approximations above, the equations (3.4) y (3.5) are exactly the optimality conditions of the continuous time problem

$$\begin{aligned} &\max_{\{c_t\}} \int_0^\infty e^{-\rho t} U(c_t) dt \\ &\text{subject to (3.5), and given } k_0, \end{aligned} \tag{3.6}$$

and obtained from direct application of optimal control theory in problem (3.6).

Beyond the precision of the approximation, the policy function of the discrete time version of the problem can be quite different from the policy function of the continuous time version. We can consider as an example the analytical case, that is, the case of full-depreciation (δ=1), and logarithmic utility, $U(c) = \log c$. We further retain the assumption of a cuasi Cobb-Douglas production function in k, that is, $f(k_t) = A\, k_t^{\alpha}$. Under such a parameterization, in the discrete time version of the model, it is immediate to obtain the policy function, which is of the form:

$$c_t = (1-\alpha\beta)Ak_t^{\alpha}. \tag{3.7}$$

On the other hand, by implementing a linear approximation around the steady state solution to the optimality conditions of the discrete time problem, the policy function would be

$$\begin{aligned} &c_t = c_{ss} + \left(\tfrac{1}{\beta} - \mu_2\right)(k_t - k_{ss}), \\ &\text{where } k_{ss} = (A\alpha\beta)^{\frac{1}{1-\alpha}},\ c_{ss} = (1-\alpha\beta)Ak_{ss}^{\alpha},\ \mu_2 = \frac{\left(\frac{1+\alpha^2\beta}{\alpha\beta}\right) - \left[\left(\frac{1+\alpha^2\beta}{\alpha\beta}\right)^2 - 4\frac{1}{\beta}\right]^{1/2}}{2}. \end{aligned} \tag{3.8}$$

Such a policy function is obtained by imposing the stability condition that cancels the eigenvector associated to the unstable eigenvalue of the dynamical system formed by the linear approximation around the steady state of the aggregate resource constraint and the Euler equation. Also, notice that the solution to the Ramsey problem for the feasible parameter space is of the saddle form, that is, it is a determinate solution.

Finally, by implementing a linear approximation around the steady state solution to the optimality conditions of the continuous time problem, the policy function obtained from eliminating the unstable manifold is:

$$\begin{aligned} &c_t = c_{ss} - \tfrac{h}{\mu_2}(k_t - k_{ss}) \\ &\text{where } k_{ss} = (A\alpha\beta)^{\frac{1}{1-\alpha}},\ c_{ss} = (1-\alpha\beta)A\, k_{ss}^{\alpha},\ \ h = \tfrac{1+\rho-\alpha}{\alpha}(1-\alpha)(1+\rho), \\ &\qquad \mu_2 = \tfrac{\rho - \left[\rho^2+4h\right]^{1/2}}{2}. \end{aligned} \tag{3.9}$$

Figure 4 depicts all three representations of the policy function in a phase diagram. The differences between the solutions in discrete vs continuous time are apparent. In particular, it is shown that at a given volatility in the state variable, the control variable fluctuates more in continuous time than in discrete time.

4 Indeterminacy of equilibria in continuous and discrete time

The dynamic properties in discrete time versions of continuous time models can fundamentally differ, particularly when the time domain at which agents take decisions differs. Convergence speeds to long-run equilibrium significantly differ between continuous and discrete time versions, the shorter the model period. This section builds upon Fernández et al. (2012) to show that transitional dynamics of pollution differ significantly, the model being written in either continuous or discrete time, and in the latter case with the amplitude of the model period.

4.1 A model of the Environmental Kuznets Curve

Fernández et al. (2012) studies the existence of an Environmental Kuznets Curve (EKC), the hypothesis that with development pollution goes up first and then down, associated to a neoclassical growth model with a pollution externality. The pollution externality goes in the utility function of the households, and it is shown that the non-separability in households preferences for consumption and pollution is crucial for the indeterminacy result to arise. Moreover, it is necessary for indeterminacy that the concern for the environment is large enough. Thus, both non-separability and enough environmentalism are needed for an Environmental Kuznets Curve pattern to be present.

The EKC result implies that economic growth could be compatible with environmental improvement if appropriate policies are taken. But before adopting a policy, it is important to understand the nature and causal relationship between economic growth and environmental quality, where the key question is whether economic growth can be part of the solution rather than the cause of environmental problem. In this section we sketch the environment in Fernández et al. (2012) and their main result, with a focus on the fact that the predictions of the theoretical model vary substantially with the length of the model period which is typically large particularly when analyzing climate issues in economic models.

In the model economy there is a continuum of identical competitive firms that maximize profits from operating a neoclassical technology. Thus,

$$\begin{aligned}&\max_{\{n_t,k_t\}} \quad y_t - \omega_t n_t - r_t k_t - \tau_P k_t\\ &\text{subject to} \quad y_t = A k_t^{\alpha} n_t^{1-\alpha}, \quad \alpha \in (0,1),\end{aligned}$$

with A being the state of technology, y_t the aggregate output, and k_t, n_t the two production factors: capital and labor. Firms rent capital from households at the interest rate r_t, pay wages w_t on labor, and pay a constant pollution tax τ_P on the level of the capital stock. Moreover, notice that with the presence of the pollution externality it turns out that equilibria of the representative agent economy is no longer Pareto optimal (as the solution of the social planner is). Therefore, we have to specify the market economy, and as a consequence, deal with the determination of market prices in an environment here with continuum of identical firms and identical households. In addition, the existence of the externality may preclude the property that equilibria are (at least locally) uniquely determined by preferences and technology, and therefore determinate, in the sense of being isolated from their neighbours.

Environmental pollution, P_t, is a side product of the capital stock used by the firms, but can be reduced by means of abatement activities made by the government, z_t

$$P_t = \frac{k_t^{\chi_1}}{z_t^{\chi_2}}, \quad \chi_1,\ \chi_2 > 0. \tag{4.1}$$

The households solve

$$\begin{aligned}
&\max_{\{c_t, n_t\}} \int_0^\infty e^{-\rho\, t} \left[\frac{\left(c_t P_t^{-\eta}\right)^{1-\sigma} - 1}{1-\sigma} - \gamma h_t \right] dt \ , \quad \gamma,\ \sigma > 0, \\
&\text{subject to} \quad c_t + \dot{k}_t + \delta k_t = (1-\tau)(\omega_t h_t + r_t k_t) + T_t, \quad \text{given}\, k_0,
\end{aligned}$$

where c_t is private consumption, P_t aggregate pollution, and h_t the labor supply.[1] σ is the inverse of the intertemporal elasticity of substitution of consumption, η is the weight of pollution in utility and γ the marginal disutility of work. Households receive income from labor and capital, that can be used to consume, save, and pay taxes at a constant rate $\tau \in (0,1)$ on the two sources of income. Finally, households receive lump-sum transfers from the government, T_t.

The problem faced by the households in the discrete time version of the economy is:

$$\begin{aligned}
&\max_{\{c_t, n_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \left[\frac{\left(c_t P_t^{-\eta}\right)^{1-\sigma} - 1}{1-\sigma} - \gamma n_t \right] \\
&\text{subject to} \quad c_t + k_{t+1} - (1-\delta) k_t = (1-\tau)(\omega_t n_t + r_t k_t) + T_t, \quad \text{given}\, k_0.
\end{aligned}$$

The Government sector in its turn chooses an income tax rate τ and an environmental tax

[1] We assume indivisible labor as in Hansen (1985). In equilibrium, $h_t = n_t$.

τ_P, and it uses these revenues to finance abatement activities, z_t and transfers to households, T_t, balancing its budget every period. Thus, the instantaneous government budget constraint is:

$$T_t + z_t = \tau\left(\omega_t n_t + r_t k_t\right) + \tau_P k_t \ , \tag{4.2}$$

where,

$$z_t = \phi\left[\tau\left(\omega_t n_t + r_t k_t\right) + \tau_P k_t\ \right]\ , \quad \phi \in [0,1], \tag{4.3}$$

ϕ being the ratio that defines the allocation of government spending to abatement activities.

4.2 Transitional dynamics

It can be shown that the dynamics of the economy in continuous time is described by

$$\begin{bmatrix} \dot{k}_t \\ \dot{\lambda}_t \end{bmatrix} = \begin{bmatrix} \mu_1 & \Omega \\ 0 & \mu_2 \end{bmatrix} \begin{bmatrix} k_t - k_{ss} \\ \lambda_t - \lambda_{ss} \end{bmatrix}, \tag{4.4}$$

whereas in discrete time it is

$$\begin{bmatrix} k_{t+1} - k_{ss} \\ \lambda_{t+1} - \lambda_{ss} \end{bmatrix} = \begin{bmatrix} \widetilde{\mu}_1 & \widetilde{\Omega} \\ 0 & \widetilde{\mu}_2 \end{bmatrix} \begin{bmatrix} k_t - k_{ss} \\ \lambda_t - \lambda_{ss} \end{bmatrix}, \tag{4.5}$$

where $\lambda(t)/\lambda_t$ is the co-state/multiplier associated to the household's budget constraint. Both dynamic systems are a linear approximation around the steady state, and therefore, $\{\mu_1, \mu_2, \Omega\}$ and $\{\widetilde{\mu}_1, \widetilde{\mu}_2, \widetilde{\Omega}\}$ are non-linear functions of structural parameters of the model.

As far as the transition matrixes are triangular, the elements in the main diagonal are the eigenvalues of the dynamical systems. Since one of the variables in the system is predetermined (k_t) and the other is free (λ_t), indeterminacy of equilibria arises only when the two roots, $\{\mu_1, \mu_2\}$ have negative real parts or $\{\widetilde{\mu}_1, \widetilde{\mu}_2\} \in (0,1)$. Fernandez et al. (2012) show that indeterminacy will arise if and only if $\sigma + (\xi_1 - \xi_2)\eta(1-\sigma) < 0$. Under indeterminacy, they show that when the economy is initially placed on the steady state and the agents eventually coordinate to choose a level of labor above its steady state value, the capital stock begins to increase in the following period and continues rising for several periods, but at an ever-decreasing rate; after reaching the turning point, the capital stock begins to decrease towards the steady state. The same behavior will be exhibited by labor, output and abatement activities. The pollution exhibits an overshooting in the first periods of the transition, leading to an inverted-U shaped pattern, the same pattern followed by the CO2 emissions by some developed economies.

To compute the dynamic response of pollution we do the following:

Step 1: Given structural parameters we solve for the systems (4.4) and (4.5) above

$$\text{Continuous time:} \begin{cases} \lambda_t = \lambda_{ss} + e^{\mu_2 t} \left(\lambda_0 - \lambda_{ss}\right), \quad \text{where 'ss' denotes steady state} \\ k_t = k_{ss} + e^{\mu_1 t} \left(k_0 - k_{ss}\right) + \frac{\Omega}{\mu_2 - \mu_1} \left(\lambda_0 - \lambda_{ss}\right) \left(e^{\mu_2 t} - e^{\mu_1 t}\right), \end{cases}$$

$$\text{Discrete time:} \begin{cases} \lambda_t = \lambda_{ss} + \widetilde{\mu}_2^t \left(\lambda_0 - \lambda_{ss}\right), \\ k_t = k_{ss} + \widetilde{\mu}_1^t \left(k_0 - k_{ss}\right) + \frac{\widetilde{\Omega}}{\widetilde{\mu}_2 - \widetilde{\mu}_1} \left(\lambda_0 - \lambda_{ss}\right) \left(\widetilde{\mu}_2^t - \widetilde{\mu}_1^t\right). \end{cases}$$

Step 2: With the expressions above, together with (4.1) to (4.3) above, we obtain a path for pollution for $k_0 = k_{ss}$, and λ_0 such that initial employment is $h_0 = n_{ss}$. Indeed, transition paths to the long-run indeterminate equilibrium can be indexed by the initial conditions in the control variable employment.

Figures (5(a)) to (5(c)) show the transitional dynamics for pollution in the theoretical economy with parameters chosen for a model period of a quarter, a year and five years. The figures illustrate that overshooting is bigger the smaller the model period, and the speed of convergence is slower in the discrete versus the continuous version of the model.

5 A continuous time model with time-to-build

Continuous time methods in economic dynamics have proven useful to give rise to substantial progress in modern growth theory and business cycle theory. The advantages of continuous time modelling are mainly technical, as far as continuous time systems turn out to be more tractable from the point of view of mathematical convenience. However, there is a crucial limitation of continuous time representations which is related to bringing the model to the data and the estimation of the structural parameters of dynamic models. An important subset of those parameters is key for economic policy design and evaluation.

Several approaches have been taken in the literature to approximate continuous-time systems by discrete time systems. Also, inference in continuous time has developed substantially in recent years. The goal of this section is to illustrate the potential of using optimal control theory with delays to bridge the gap between continuous and discrete time representations in economic dynamics. We characterize the link between the two representations with a simple example of a growth economy with time–to–build.

5.1 The environment

Let us primarily recover the centralized description of the economy as in Section 3, as we are under the conditions that the market allocation can be achieved by a social planner. Let us assume that time is continuous and introduce a simple time–to–build technology in an otherwise standard one-sector growth model. For simplicity, all variables are in per capita terms. Let $d > 0$ be the planned horizon of an investment project, *i.e.* the time–to–build delay. The technology to produce one unit of the investment good available at time $t + d$ requires a flow of $\frac{1}{d}$ units of the final good in the time interval $[t, t + d]$. Consequently, the only relevant decision at time t is the amount of *planned investment* $i(t)$, which will become operative at time $t + d$.

The stock of *planned capital* at time $t \geq -d$ is given by

$$k(t) = k(-d) + \int_{-d}^{t} i(s)\ \mathrm{d}s. \tag{5.1}$$

The implicit assumption of zero depreciation makes $i(t)$ to be net investment. By definition of $i(t)$, $k(t)$ becomes operative at time $t + d$. Initial conditions need to be specified: $k(-d) = \bar{k} > 0$ and $i(t) = i_0(t) \geq 0$ for all $t \in [-d, 0[$. Consequently, $k(t) = k_0(t)$ for all $t \in [-d, 0]$ is computed using (5.1).

Final output is produced using a standard neoclassical technology $f(k)$, assumed to be C^2, increasing and concave for $k > 0$ and verifying Inada conditions [cf. Inada (1963) and Uzawa (1963)]. Operative capital at time t was already planned at time $t - d$, implying that production at time t is $f(k(t - d))$.

The production of the final good is allocated to consumption $c(t)$ and to net investment expenditures $x(t)$. At time $t \geq 0$, the amount of the final good employed in the production of investment goods is given by

$$x(t) = \frac{1}{d} \int_{t-d}^{t} i(s) \, \mathrm{d}s. \tag{5.2}$$

It corresponds to investment expenditures associated to all active investment projects. Under these assumptions, the feasibility constraint for $t \geq 0$ takes the following form:

$$f\left(k(t-d)\right) = c(t) + x\left(t\right). \tag{5.3}$$

5.2 The planner's problem

As in Section 3 above, in the described environment the solution of a benevolent social planner is the competitive equilibrium allocation. Let such a planner maximize the utility of the representative household

$$\max \int_{0}^{\infty} u\left(c\left(t\right)\right) \ \mathrm{e}^{-\rho t} \tag{P}$$

subject to (5.3) and

$$\dot{x}(t) = \frac{1}{d}\left(i(t) - \ i(t-d)\right), \tag{5.4}$$

$$\dot{k}(t) = i\left(t\right). \tag{5.5}$$

Constraints (5.4) and (5.5) result from time differentiation of (5.2) and (5.1), respectively. The initial conditions are $x\left(0\right) = x_0 = \frac{1}{d} \int_{-d}^{0} i_0(s) \ \mathrm{d}s$, $k\left(t\right) = k_0\left(t\right)$ and $i(t) = i_0(t)$ for all $t \in [-d, 0[$, as specified previously. The instantaneous utility function $u(t)$ is C^2, increasing and concave for $c > 0$, and verifies Inada conditions.

Using optimal control theory with delays [cf. Kolmanovskii and Myshkis (1998), Boucekkine et al. (2005) or Bambi et al. (2014)], necessary first-order-conditions for this problem are

$$u'(c\left(t\right)) = \phi\left(t\right), \tag{5.6}$$

$$\lambda(t) + \frac{1}{d}\mu(t) = \frac{1}{d}\mu(t+d)\ \mathrm{e}^{-\rho d}, \tag{5.7}$$

$$-\phi(t+d) f'(k(t))\ \mathrm{e}^{-\rho d} = \dot{\lambda}(t) - \rho\lambda(t), \tag{5.8}$$

$$\phi(t) = \dot{\mu}(t) - \rho\mu(t), \tag{5.9}$$

and the transversality conditions

$$\lim_{t\to\infty} k(t)\lambda(t)\ \mathrm{e}^{-\rho t} = 0, \tag{5.10}$$

$$\lim_{t\to\infty} x(t)\mu(t)\ \mathrm{e}^{-\rho t} = 0. \tag{5.11}$$

The Lagrange multiplier $\phi(t)$ is associated to constraint (5.3), and the co-states $\lambda(t)$ and $\mu(t)$ are associated to the states $k(t)$ and $x(t)$, respectively. Advanced terms appearing in (5.7) and (5.8), related to the delays in (5.4) and (5.5), make explicit the trade-offs. Marginal investment at time t has three different effects on utility. Firstly, it increases planed capital, which marginal value is $\lambda(t)$. Second, it rises investment expenditures, with marginal costs $\frac{\mu(t)}{d}$. Finally, when the project will be finished at $t+d$, investment expenditures will end.

5.3 Discrete Time as a Representation of Continuous Time

In this section we establish the correspondence between the proposed continuous time model with a time–to–build delay in Section 5.2 and the discrete time representation of the neoclassical growth model in Section 3. Let us assume that the initial function $i_0(t)$ is piecewise continuous, and that feasible trajectories $i(t)$, for $t \geq 0$, belong to the family of piecewise continuous functions.

Proposition 1. *Under $d = 1$, the optimal conditions (5.6) to (5.11) of problem (P) become*

$$k(t) - k(t-1) = f(k(t-1)) - c(t) \tag{5.12}$$

$$\frac{u'(c(t))}{u'(c(t+1))} = \beta\left(1 + f'(k(t))\right), \tag{5.13}$$

where $\beta \equiv e^{-\rho}$.

Proof. From (5.1) and (5.2), under $d = 1$, we get $x(t) = k(t) - k(t-1)$. The feasibility constraint

(5.12) results from substituting the relation between x and k on equation (5.3). Differentiating (5.7), substituting $\dot{\lambda}$ and $\dot{\mu}$ by (5.8) and (5.9), after some rearrangements, we get (5.13). $\square$

The equilibrium path of the neoclassical growth model is indeed represented by (5.12) and (5.13) for given initial conditions. Notice the correspondence with system (3.1) and (3.3) in Section 3 under the assumption here of zero depreciation, that is $\delta = 0$.

Corollary 2. *The steady state solution of (5.12) and (5.13) is saddle-path stable for $t \geq 0$.*

Corollary 2 implies that for every $s \in [0, 1)$, the optimal sequence $\{c_{s+i}, k_{s+i}\}$, for $i = \{0, 1, 2, 3...\}$, is the solution of the discrete time neoclassical Ramsey growth model of Section 3 with zero depreciation., given $k(-1) = k_0(-1)$. However, in continuous time with delay $d = 1$, the steady state solution involves the solution path for all $s \in [0, 1)$, which depends on the boundary function $k_0(t)$, for $t \in [-1, 0)$ defining initial conditions [cf. Collard et al. (2008)].

Corollary 3. *Under $d = 1$ and $k(t) = k_0 > 0$ for $t \in [-1, 0)$, the optimal solution $k(t), c(t)$ of problem (P) is constant in the interval $[i - 1, i)$ for $i = \{1, 2, 3...\}$ and it corresponds to the stable brand of the discrete problem in Proposition 1.*

Corollary 3 state the dynamic properties of the neoclassical growth model in its correspondence with the continuous time representation with delays. All in all, the example illustrates on the particular form of a delay in the continuous time model that the discrete time model imposes. The recent literature on continuous time with delays should help to take advantage of the analytical tractability of models in continuous time while providing precise quantitative statements about the issues of interest.

6 Concluding remarks

In this paper, we have explored the differences arising from modelling time as discrete or continuous. This has been done in the basic framework of dynamic macroeconomic models and focusing on appropriate approximation, dynamic indeterminacy and delays. We have shown that the differences between continuous vs discrete representations arise from investment decisions at time t, that become productive at a time that depends on the model period. Agents are then committed to their decisions until the period when the return of their investments is realized. This modifies not only the structure and the solution of the models, but also the economic interpretation of the results. The recent literature on continuous time models with delays should help to bridge the gap between continuous and discrete time representations in economic dynamics.

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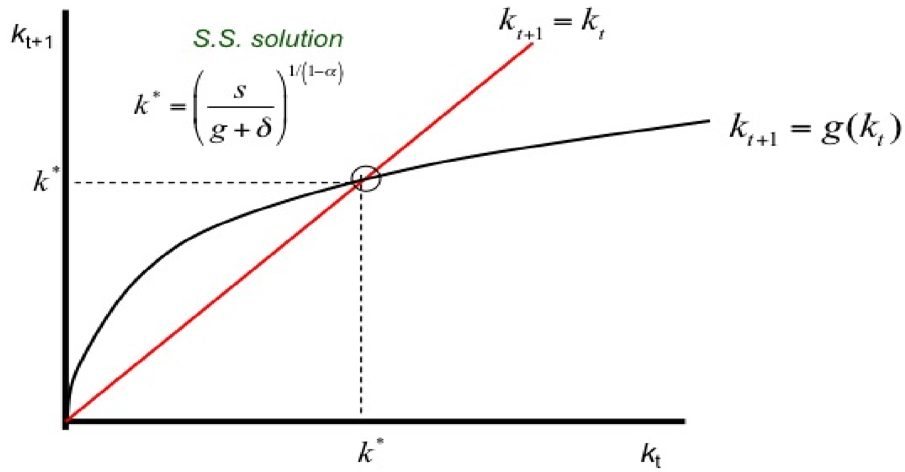


Figure 1: The Steady State solution and the dynamics of the Solow Growth Model in discrete time, with $k_{t+1} = \eta\,[(1-\delta)\,k_t + s\,k_t^{\alpha}] \equiv g(k_t)$, where $\eta = 1/(1+g)$ and $k^* = k_{ss}$.

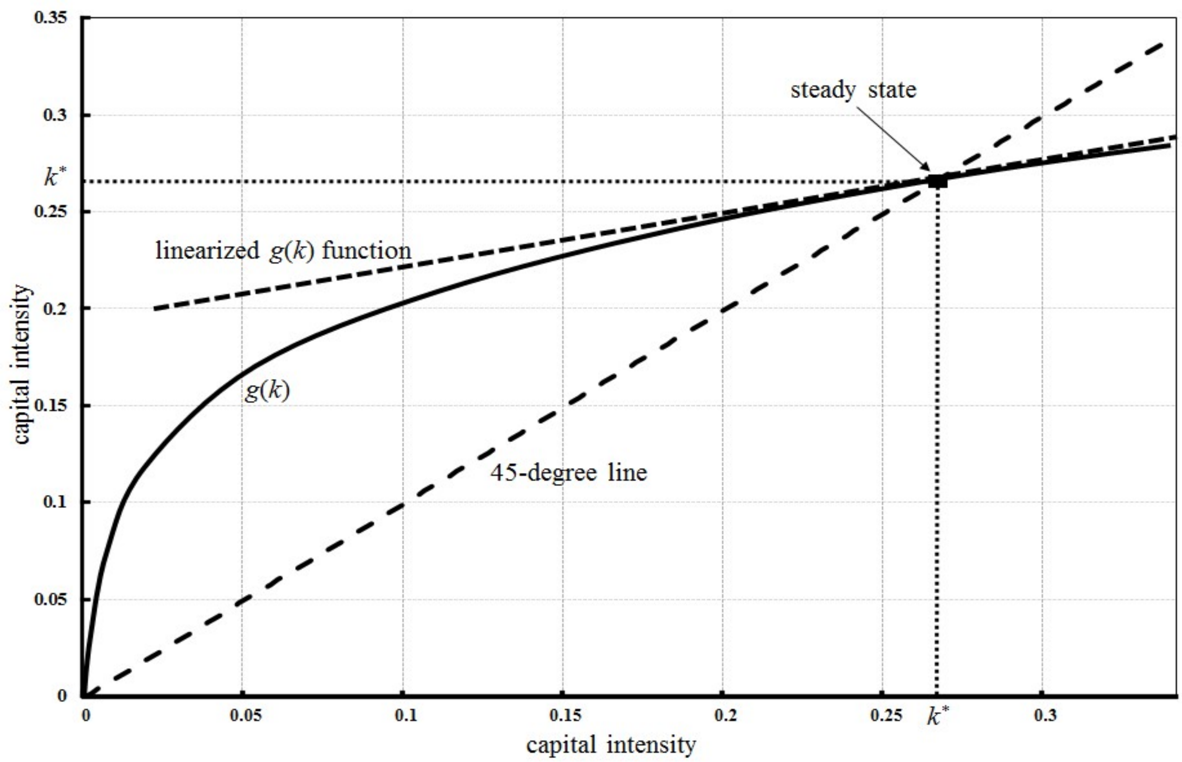


Figure 2: The policy function $g(k)$, with $k_{t+1} = \eta\,[(1-\delta)\,k_t + s\,k_t^{\alpha}] \equiv g(k_t)$, where $\eta = 1/(1+g)$ and $k^* = k_{ss}$, and a linear approximation.

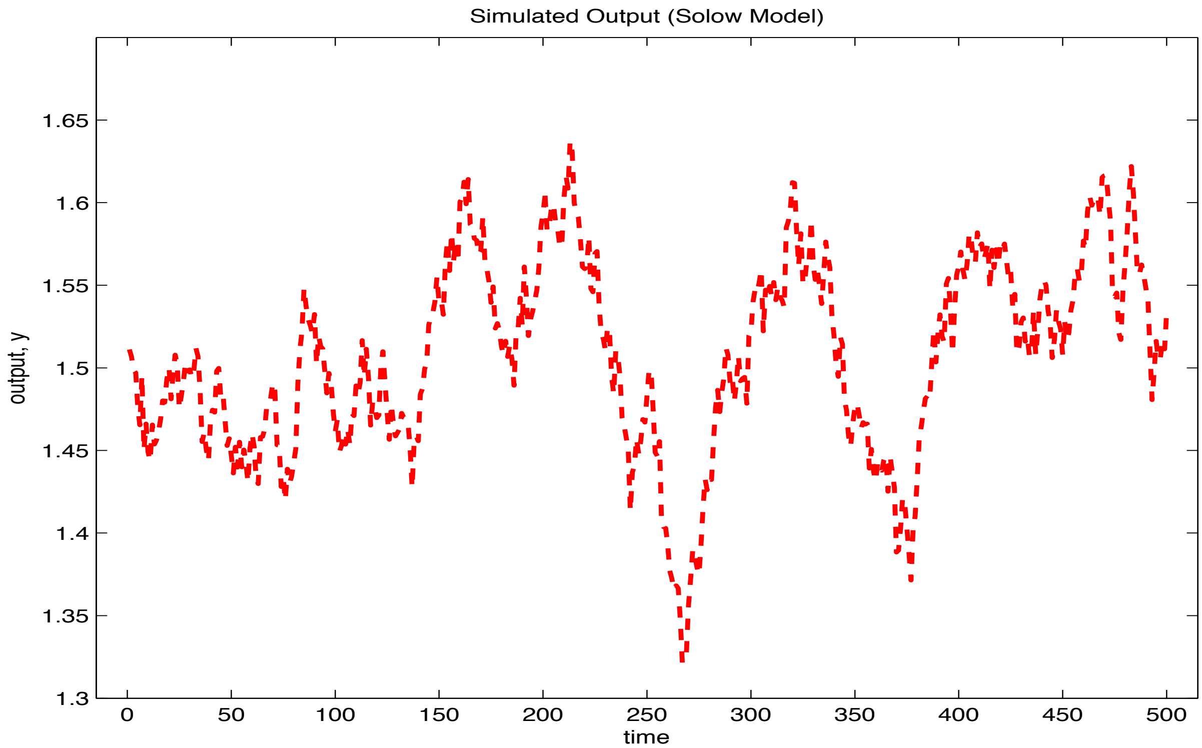


Figure 3: An example of output fluctuations in the Stochastic Solow model: output values around the deterministic steady state over 500 periods. Parameters: $\alpha = 0.36$, $g = 0.02$, $\delta = 0.1$, $s = 0.25$, $\rho = 0.95$, $\sigma_\varepsilon = 0.01$.

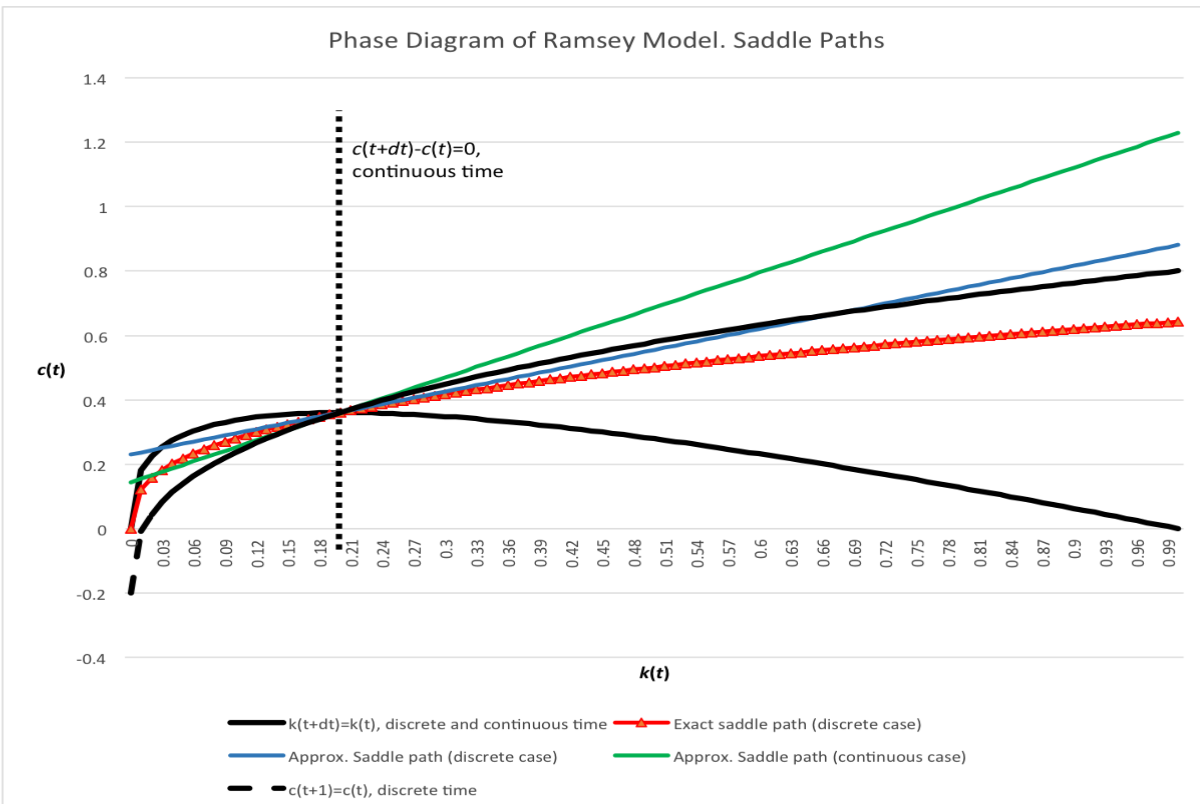
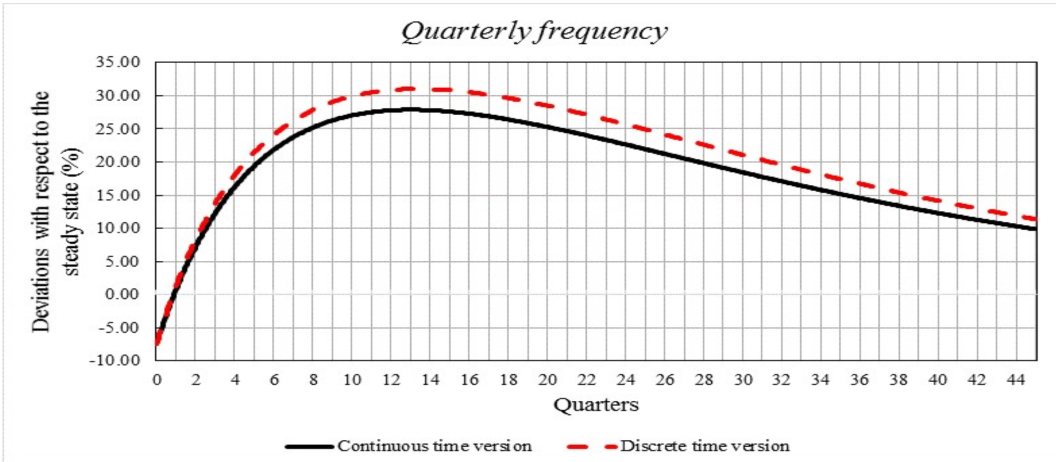
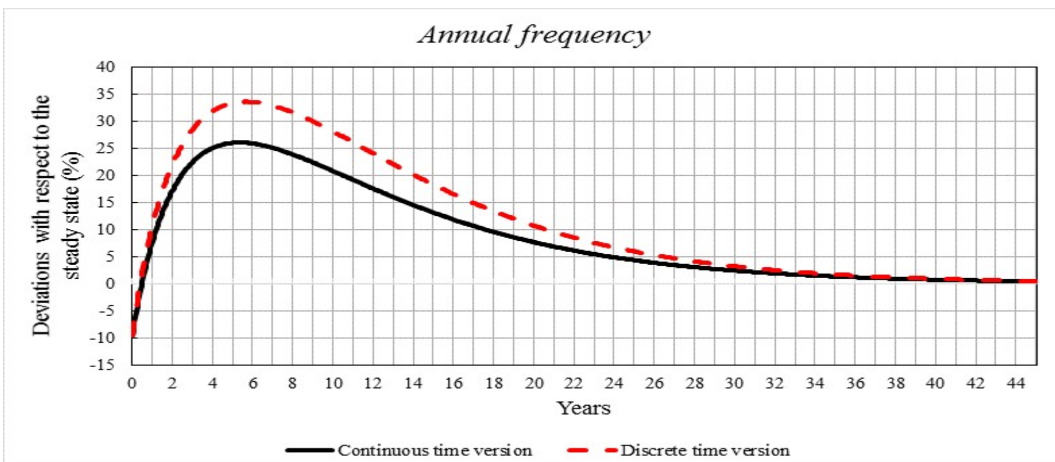


Figure 4: Saddle Paths of the Ramsey model

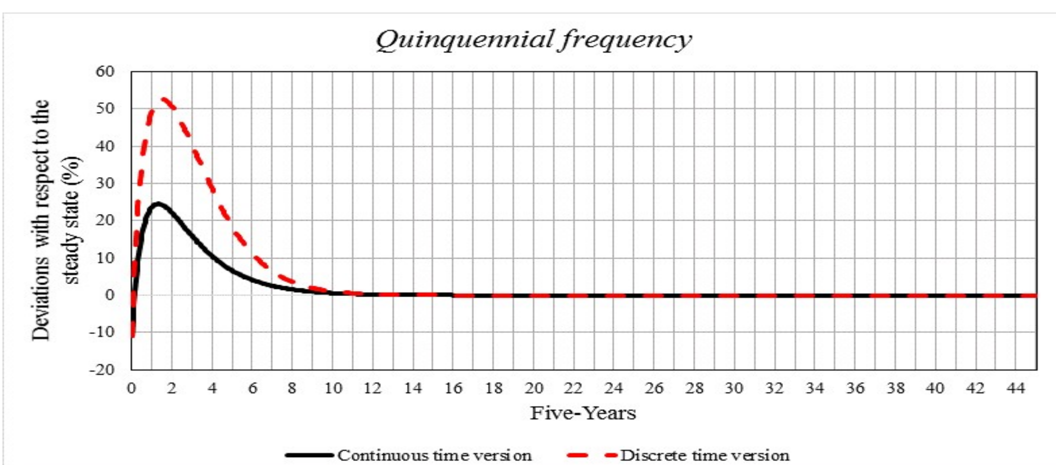
Pollution paths: Indeterminacy case with $n_0 > n_{ss}$



(a) Parameters quarterly: $\alpha = 0.33, \chi_1 = 1.3, \chi_2 = 0.6, \tau_P = 4\%, \tau = 20\%, \sigma = 2.5, \eta = 3.5, A = 1, \phi = 0.1, \delta = 0.025$ (10% per year), $\rho = 0.01$ (4% per year); γ is chosen to match n_{ss}, and $n_0 = 0.4$.



(b) Parameters yearly: parameters above, but for $\delta = 0.1$ (10% per year), $\rho = 0.04$ (4% per year)



(c) Parameters five-year: parameters above, but for $\delta = 0.5$ (10% per year), $\rho = 0.2$ (4% per year)